MATH 121A Prep: Subspaces

1. Let $\vec{v} \in \mathbb{R}^n$, and define $span(\vec{v}) = \{c\vec{v} : c \in \mathbb{R}\}$. Show that $span(V)$ is a subspace of $\mathbb{R}^n$.

> **Solution:** We need to show three things: this set is non-empty, closed under addition, and closed under scalar multiplication.
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> Non-empty: $\vec{v} = 1\vec{v} \in span(\vec{v})$ so non-empty.
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> Closed Under Addition: Suppose $\vec{u}, \vec{w} \in span(\vec{v})$. Then $\vec{u} = c_1\vec{v}$ and $\vec{w} = c_2\vec{v}$ for some real numbers c_1 and c_2. Then $\vec{u} + \vec{w} = c_1\vec{v} + c_2\vec{v} = (c_1 + c_2)\vec{v} \in span(\vec{v})$.
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> Closed Under Scalar Multiplication: Let $\vec{u} \in span(\vec{v})$ and d a real number. Then $\vec{u} = c\vec{v}$ for a real number c, and so $d\vec{u} = dc\vec{v} \in span(\vec{v})$
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> Thus this subset is a subspace.

2. Explain why $\{(x, y) \in \mathbb{R}^2 : xy = 0\}$ is not a subspace of $\mathbb{R}^2$.

> **Solution:** The vectors $\begin{bmatrix}1\\0\end{bmatrix}$ and $\begin{bmatrix}0\\1\end{bmatrix}$ both belong to this subset. Their sum is $\begin{bmatrix}1\\1\end{bmatrix}$ but for this vector $xy = 1 \neq 0$ so this subset is not closed under addition and therefore not a subspace.

3. Let A be an $m \times n$ matrix and $W = \{A\vec{v} : \vec{v} \in \mathbb{R}^n\}$. Is W a subset of $\mathbb{R}^n$ or $\mathbb{R}^m$? Show that W is in fact a subspace.

> **Solution:** Since A is an $m \times n$ matrix it maps vectors in $\ltimes$ to vectors in $\mathbb{R}^m$. Therefore $A\vec{v} \in \mathbb{R}^m$ for $\vec{v} \in \mathbb{R}^n$ so W is a subset of $\mathbb{R}^m$.
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> To show this is a a subspace:
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> Non-empty: $\vec{0} = A\vec{0} \in W$ so this is non-empty.
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> Closed Under Addition: Let $\vec{w_1}$ and $\vec{w_2} \in W$. Then $\vec{w_1} = A\vec{v_1}$ and $\vec{w_2} = A\vec{v_2}$ for some vectors $\vec{v_1}, \vec{v_2} \in \mathbb{R}^n$. Then
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> $$\vec{w_1} + \vec{w_2} = A\vec{v_1} + A\vec{v_2} = A(\vec{v_1} + \vec{v_2}) \in W$$
>
> as desired.
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> Closed Under Scalar Multiplication: Let $\vec{w} \in W$ and $c \in \mathbb{R}$. Then $\vec{w} = A\vec{v}$ for some vector $\vec{v} \in \mathbb{R}^n$. Now
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> $$c\vec{w} = cA\vec{v} = A(c\vec{v}) \in W$$
>
> Therefore W is a subspace of $\mathbb{R}^m$.